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RATIO-DEPENDENT PREDATOR–PREY MODEL WITH STAGE STRUCTURE AND TIME DELAY

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Ratio-dependent predator–prey models are favored by many animal ecologists recently as more suitable ones for predator–prey interactions where predation involves searching process. In this paper, a ratio-dependent predator–prey model with stage structure and time delay for prey is proposed and analyzed. In this model, we only consider the stage structure of immature and mature prey species and not consider the stage structure of predator species. We assume that the predator only feed on the mature prey and the time for prey from birth to maturity represented by a constant time delay. At first, we investigate the permanence and existence of the proposed model and sufficient conditions are derived. Then the global stability of the nonnegative equilibria are derived. We also get the sufficient criteria for stability switch of the positive equilibrium. Finally, some numerical simulations are carried out for supporting the analytic results.

Keywords: Ratio-dependent; stage structure; time delay; characteristic equation; globally asymptotical stability.

1. Introduction

In the natural world, almost all animals have the stage-structure of immature and mature. Hence, it is of ecological importance to investigate the effects of such a

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subdivision on the interaction of species. In recent years, stage-structured models have been received great attention. In [1], Aiello and Freedman proposed and invested a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model. This model assumed an average age to maturity which appears as a constant time delay reflecting a delayed birth of the immatures and a reduced survival of the immatures to their maturity. The model takes on the following form

$$\begin{cases} \dot{x}_i(t) = \alpha x_m - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) = \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2, \end{cases} \tag{1.1}$$

where $x_i(t)$ denotes the immature population density, $x_m(t)$ represents mature population density, $\alpha > 0$ represents the birth rate, $\gamma > 0$ is the immature death rate, $\beta > 0$ is the mature death and overcrowding rate, τ is the time to maturity. The term $\alpha e^{-\gamma\tau} x_m(t-\tau)$ represents the immatures who were born at time $t-\tau$ and survive at time t (with the immature death rate γ), and therefore represents the transformation of immaturity to maturity. Following the way of Aiello and Freedman, many authors studied some different kinds of stage-structured models and some significant work has been carried out (see, for example, [9, 10, 13, 14, 19, 24, 28]).

A large body of existing prey–predator models have appeared in the literature, which assumed that the per capita rate of predation depends on the prey numbers only. That is not reasonable especially when predators have to search, share or compete for food. In 1959 and 1966, Holling proposed three types of functional response and proved that the functional response played an important role in predator–prey systems [17, 18]. A more suitable general predator–prey model should be based on the "ratio-dependent" theory. The per capita predator growth rate should be a function of the ratio of prey to predator abundance. Moreover, the number of predators is relative to prey number and it often changes slowly because of the competition among the predators, and the per capita rate of predation depending on the numbers of both prey and predator, most probably and simply on their ratio [27]. These hypotheses are strongly supported by numerous field and laboratory experiment and observations [2–4, 16]. Hence, ratio-dependent predator–prey models with Michaelis Menten type functional response have received great attention [25, 27]. Recently, ratio-dependent prey-dependent predator–prey models have been studied by several researchists (see, for example, [3–5, 12, 15])

The predator–prey systems with stage-structure are very important and have been studied by many authors. In [22], Shi and Chen considered the following model,

$$\begin{cases} \dot{x}_1(t) = \alpha x_2(t) - d_1 x_1(t) - \alpha e^{-d_1\tau} x_2(t-\tau), \\ \dot{x}_2(t) = \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \dfrac{p x_2(t) y(t)}{m y(t) + x_2(t)}, \\ \dot{y}(t) = -d_4 y(t) + \dfrac{f p x_2(t) y(t)}{m y(t) + x_2(t)}, \end{cases} \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ denote the immature and mature prey population densities at time t, $y(t)$ is the density of predator population at time t, $\alpha > 0$ is the birth rate and transformation rate of immature population, $d_1 > 0$ is the death rate of immature population, $d_2 > 0$ is the death rate of mature population, $d_3 > 0$ is the intra-specific competition rate of mature population, $p > 0$ is capturing rate, $m > 0$ is half capturing saturation constant, $d_4 > 0$ is the predator death rate, f is the conversion rate for predation. System (1.2) assumes the prey population have stage-structure and only mature individuals are consumed by the predator. This seems reasonable for a number of mammals. As is common, the dynamics-eating habits, susceptibility to predators, are often quite different in these two sub-populations. In the natural world, when the immature preys conceal in the mountain cave and are raised by their parents, they do not necessarily go out to seek food, then the rate they are attacked by the predators can be ignored. In [22], Shi and Chen discussed the stability of equilibria and the effect of impulsive interruption on the original model.

In this paper, we assume that the present number of the predator affects instantaneously the number of the maturity prey, but that the growth of the predator is influenced by the amount of the maturity prey in the past [25]. To this end, we consider the following more reasonable integro-differential equations with time delay

$$\begin{cases} \dot{x_1}(t) = \alpha x_2(t) - d_1 x_1(t) - \alpha e^{-d_1\tau} x_2(t-\tau), \\ \dot{x}_2(t) = \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \dfrac{p x_2(t) y(t)}{m y(t) + x_2(t)}, \\ \dot{y}(t) = -d_4 y(t) + h \displaystyle\int_{-\infty}^{t} \frac{\delta x_2(s) y(s)}{m y(s) + x_2(s)} e^{-\delta(t-s)} ds, \end{cases} \tag{1.3}$$

the exponential weight function satisfies

$$\int_{-\infty}^{t} \delta e^{-\delta(t-s)} ds = \int_{0}^{\infty} \delta e^{-\delta u} du = 1. \tag{1.4}$$

More precisely, the number of predators grows depending on the weight-averaged time of the Michaelis-Menten function of $x_2(t)$ over the past by means of the function $z(t)$ given by the integral

$$z(t) = \int_{-\infty}^{t} \frac{\delta x_2(s) y(s)}{m y(s) + x_2(s)} e^{-\delta(t-s)} ds. \tag{1.5}$$

Clearly, this assumption implies that the influence of the past fades away exponentially and the number $1/\delta$ might be interpreted as the measure of the influence of the past. So, the smaller the $\delta > 0$, the longer the interval in the past in which the values of $x_2(t)$ are taken into account [7, 11, 20].

The integro-differential system (1.3) can be transformed [11, 20] into the system of differential equations on the interval $[0, \infty)$

$$\begin{cases} \dot{x_1}(t) = \alpha x_2(t) - d_1 x_1(t) - \alpha e^{-d_1\tau} x_2(t-\tau), \\ \dot{x}_2(t) = \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \dfrac{p x_2(t) y(t)}{m y(t) + x_2(t)}, \\ \dot{z}(t) = \dfrac{\delta x_2(t) y(t)}{m y(t) + x_2(t)} - \delta z(t), \\ \dot{y}(t) = -d_4 y(t) + h z(t). \end{cases} \tag{1.6}$$

The relationship between systems (1.3) and (1.6) as follows: if $(x_1(t), x_2(t), y(t)) : [0, \infty) \to R^3$ is the solution of (1.3) corresponding to continuous and bounded initial function $(\bar{x}_1(t), \bar{x}_2(t), \bar{y}(t)) : [0, \infty) \to R^3$ then $(x_1(t), x_2(t), z(t), y(t)) : [0, \infty) \to R^4$ is a solution of (1.6) with $x_1(0) = \bar{x}_1(0), x_2(0) = \bar{x}_2(0), y(0) = \bar{y}(0)$ and

$$z(0) = \int_{-\infty}^{0} \frac{\delta \bar{x}_2(\tau) \bar{y}(\tau)}{m\bar{y}(\tau) + \bar{x}_2(\tau)} e^{\delta\tau} d\tau. \tag{1.7}$$

Conversely, if $(x_1(t), x_2(t), z(t), y(t))$ is any solution of (1.6) defined on the real line and bounded on $[0, \infty)$, then $z(t)$ is given by (1.5) and so $(x_1(t), x_2(t), y(t))$ satisfies (1.3).

In the next sections of this paper, we study model (1.6) with the following initial conditions:

$$\begin{cases} x_1(\theta) = \phi_0(\theta) \geq 0, \quad x_2(\theta) = \phi_1(\theta) \geq 0, \quad z = \phi_2(\theta) \geq 0, \\ y(\theta) = \phi_3(\theta) \geq 0, \quad \phi_0(0) > 0, \quad \phi_1(0) > 0, \\ \phi_2(0) > 0, \quad \phi_3(0) > 0, \quad \theta \in [-\tau, 0], \end{cases} \tag{1.8}$$

where $(\phi_0(\theta), \phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], R_4^+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R_4^+, where $R_4^+ = \{(x_1, x_2, x_3, x_4) : x_i \geq 0, i = 1, 2, 3, 4\}$. For continuous of the initial conditions, we further require

$$x_1(0) = \int_{-\tau}^{0} \alpha e^{d_1 s} \phi_2(\theta) ds. \tag{1.9}$$

This paper is organized as follows. In the next section, we discuss the permanence and extinction of the system (1.6). In Sec. 3, we study the stability of one nonnegative equilibrium. The existence of Hopf bifurcation at the positive equilibrium is presented in Sec. 4. In Sec. 5, some numerical simulations are performed to illustrate the analytical results. A brief discussion is given in Sec. 6 to conclude this work.

2. Boundary Dynamics and Permanence

It is important to show the positivity and boundedness for the system (1.6) as they represent populations. Boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources. In this section, we present some basic results, such as boundedness of solutions, the permanence and extinction of the system.

As a direct corollary of [8, Lemma 1.4], we have the following lemma.

Lemma 2.1. *If* $a > 0,\ b > 0$ *and* $\dot{x}(t) \geq b - ax(t)$ *when* $t \geq 0$ *and* $x(0) > 0$, *we have* $\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}$.

If $a > 0, b > 0$ *and* $\dot{x}(t) \leq b - ax(t)$ *when* $t \geq 0$ *and* $x(0) > 0$, *we have* $\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}$.

We need the following results from [23].

Lemma 2.2. *Consider the following equation*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t),$$

where $a, b, c,$ *and* τ *are positive constants,* $x(t) > 0$ *for* $t \in [-\tau, 0]$. *We have*

(i) *If* $a > b$, *then* $\lim_{t\to+\infty} x(t) = \frac{a-b}{c}$;
(ii) *If* $a < b$, *then* $\lim_{t\to+\infty} x(t) = 0$.

Theorem 2.3. *Solutions of system* (1.6) *with initial conditions* (1.8) *and* (1.9) *are nonnegative and bounded for all* $t \geq 0$.

Proof. Firstly, we show $x_1(t) > 0, x_2(t) > 0, z(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$.

Let $(x_1(t), x_2(t), z(t), y(t))$ be a solution of system (1.6) with initial conditions (1.8) and (1.9). First we show $x_2(t) > 0$, for all $t \geq 0$. Otherwise if it is false, noting that $x_2(t) > 0$, $(-\tau \leq t \leq 0)$, $y(t) > 0$, $(-\tau \leq t \leq 0)$, then there exists a $t^{'} \geq 0$, such that $x_2(t^{'}) = 0$. Now we define $t_0 = \inf\{t > 0 | x_2(t) = 0\}$, then $t_0 > 0$, and from the second equation of system (1.6), we have

$$\dot{x}_2(t_0) = \begin{cases} \alpha e^{-d_1\tau}\phi_1(t_0 - \tau), & 0 \leq t_0 \leq \tau, \\ \alpha e^{-d_1\tau} x_2(t_0 - \tau), & t_0 > \tau. \end{cases}$$

Then $x_2(t_0) > 0$, but by the definition of t_0, $x_2(t_0) = 0$, this is a contradiction. Hence $x_2(t) > 0$, for all $t \geq 0$.

Next we will show $z(t) \geq 0$, $y(t) \geq 0$, for all $t \geq 0$.

If there exists $t^{'} \geq 0$, and denote $t_0^{'} = \inf\{t > 0 | z(t) = 0\}$, then $t_0^{'} > 0$, and from the third equation of system (1.6), we have

$$\dot{z}(t_0^{'})|_{y\geq 0} = \frac{\delta x_2(t) y(t)}{my(t) + x_2(t)} \geq 0,$$

hence, we get $z(t) \geq 0$. Similarly, we can derive $y(t) \geq 0$. By (1.9) and the first equation of system (1.6), we derive $x_1(t) = \int_{t-\tau}^{t} \alpha e^{-d_1(t-s)} x_2(s) ds > 0$ for $t \in [0, \tau]$, clearly, $x_1(t) > 0$, for all $t \geq 0$.

Thus, $x_1(t) > 0, x_2(t) > 0, z(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$.

Now we consider the boundedness of positive solutions of system (1.6).

Let $(x_1(t), x_2(t), z(t), y(t))$ be any positive solution of system (1.6) with initial conditions (1.8). Define $\rho(t) = \delta x_1(t) + \delta x_2(t) + pz(t)$. Calculating the derivative of $\rho(t)$ along positive solutions of (1.6), it follows that

$$\begin{aligned}\dot{\rho}(t) &= \alpha\delta x_2(t) - \delta d_1 x_1(t) - \delta d_2 x_2(t) - \delta d_3 x_2^2(t) - \delta pz(t),\\ &\le -A\rho(t) + \delta\alpha x_2(t) - \delta d_3 x_2^2(t),\\ &\le -A\rho(t) + \frac{\delta\alpha^2}{4d_3},\end{aligned} \tag{2.1}$$

where $A = \min\{d_1, d_2, p\}$. Therefore, by Lemma 2.1, we have

$$\lim_{t\to\infty} \rho(t) \le \frac{\delta\alpha^2}{4Ad_3}.$$

Then there exists an M_1, depending only on the parameters of system (1.6), such that for any $t > T$, we have

$$x_1(t) < M_1, \quad x_2(t) < M_1, \quad z(t) < M_1.$$

It follows from the fourth equation of system (1.6) that for $t > T$,

$$\dot{y}(t) < -d_4 y(t) + hM_1.$$

By Lemma 2.1, we have

$$\limsup_{t\to+\infty} y(t) \le \frac{hM_1}{d_4}.$$

Define $M = \max\{\frac{hM_1}{d_4}, M_1\}$. Hence, we have

$$0 \le x_1(t), x_2(t), z(t), y(t) \le M.$$

This completes the proof. □

Since the last three equations of system (1.6) have no relation to variable $x_1(t)$, we only need to investigate the following system (2.2):

$$\begin{cases}\dot{x}_2(t) = \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \dfrac{px_2(t)y(t)}{my(t) + x_2(t)},\\[2ex] \dot{z}(t) = \dfrac{\delta x_2(t)y(t)}{my(t) + x_2(t)} - \delta z(t),\\[2ex] \dot{y}(t) = -d_4 y(t) + hz(t).\end{cases} \tag{2.2}$$

In the next sections of this paper, we focus on model (2.2) with initial conditions (1.8).

First, we introduce a definition which is useful to study the permanence of the system (2.2).

Definition 2.4. (i) System (2.2) is said to be uniformly persistent if there exists a compact region $D \subset IntR_+^3$, such that every solution $X(t)$ of system (2.2) with initial conditions (1.8) eventually enters and remains in the region D.

(ii) System (2.2) is said to be impermanent if there is a positive solution $(x_2(t), z(t), y(t))$ of (2.2) satisfying

$$\min\left\{\lim_{t\to+\infty}\inf x_2(t),\ \lim_{t\to+\infty}\inf z(t),\ \lim_{t\to+\infty}\inf y(t)\right\} = 0.$$

Theorem 2.5. *If* (i) $\alpha e^{-d_1\tau} - d_2 - \frac{p}{m} > 0$ *and* (ii) $h > d_4$ *hold, then system* (2.2) *is uniformly persistent.*

Proof. Obviously, $\Omega = \{x_2(t), z(t), y(t)) \mid 0 \le x_2(t), z(t), y(t) \le M\}$ is a positively invariant set of system (2.2). Given any positive solution $(x_2(t), z(t), y(t))$ of system (2.2), we have

$$\dot{x}_2(t) > \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \frac{p x_2}{m}. \tag{2.3}$$

Consider the following auxiliary equation

$$\dot{u}(t) = \alpha e^{-d_1\tau} u(t-\tau) - d_2 u(t) - d_3 u^2(t) - \frac{p u(t)}{m}. \tag{2.4}$$

By applying Lemma 2.2 to system (2.4), when $\alpha e^{-d_1\tau} - d_2 - \frac{p}{m} \ge 0$, one obtains

$$\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-d_1\tau} - d_2 - \frac{p}{m}}{d_3},$$

by comparison, we get

$$\liminf_{t\to+\infty} x_2(t) > \frac{\alpha e^{-d_1\tau} - d_2 - \frac{p}{m}}{d_3} := \underline{x_2}.$$

Hence there is a $T > 0$, such that $x_2(t) > \frac{\underline{x_2}}{2}$, for $t > T$, and we have

$$\begin{cases} \dot{z}(t) > \dfrac{\delta\frac{\underline{x_2}}{2} y(t)}{m y(t) + \frac{\underline{x_2}}{2}} - \delta z(t), \\ \dot{y}(t) = -d_4 y(t) + h z(t). \end{cases} \tag{2.5}$$

Now, we consider the comparison equations

$$\begin{cases} \dot{u}(t) = \dfrac{\delta\frac{\underline{x_2}}{2} v(t)}{m v(t) + \frac{\underline{x_2}}{2}} - \delta u(t), \\ \dot{v}(t) = -d_4 v(t) + h u(t). \end{cases} \tag{2.6}$$

Obviously, there can exist two equilibria of (2.6): $(0,0)$ and (u^*, v^*), where $u^* = \frac{d_4}{h} v^*$, $v^* = \frac{\underline{x_2}}{2m}(\frac{h}{d_4} - 1)$.

Linearizing system (2.6) at $(0,0)$, we derive the characteristic equation of $(0,0)$ is

$$\lambda^2 + (\delta + d_4)\lambda + \delta(d_4 - h) = 0. \tag{2.7}$$

Clearly, if $h > d_4$, then (2.7) has a positive root, then the equilibrium $(0,0)$ of system (2.6) is unstable.

Linearizing system (2.6) at (u^*, v^*), we derive the characteristic equilibrium of (u^*, v^*) is

$$\lambda^2 + (\delta + d_4)\lambda + \delta d_4 \left(1 - \frac{d_4}{h}\right) = 0. \tag{2.8}$$

It is easy to see that if $h > d_4$, then (2.8) has two negative real roots, then the equilibrium (u^*, v^*) of system (2.6) is locally asymptotically stable.

Let $0 < u(t_0) < z(t_0), 0 < v(t_0) < y(t_0), t_0 > T$. If $(u(t), v(t))$ is a solution of (2.6) with initial conditions $(u(t_0), v(t_0))$ for $t_0 > T$, then $z(t) \geq u(t), \underline{y(t)} \geq v(t)$ for $t > t_0$. For (2.6), if there exists a solution which is unbounded, say $(\overline{u(t)}, \overline{v(t)}) \to (+\infty, +\infty)$ as $t \to +\infty$, then it follows that for (2.2) there exists at least one solution, say $(x_2(t), z(t), \underline{y(t)})$, which is also unbounded provided there is a satisfying initial condition $0 < \overline{u(t_0)} < z(t_0), 0 < \overline{v(t_0)} < y(t_0)$. This contradicts the boundedness of solutions of (2.2). Hence we must have that all the solutions of (2.6) are bounded. It follows that the unique positive equilibrium (u^*, v^*) is globally asymptotically stable. Hence we have

$$\liminf_{t\to+\infty} z(t) \geq u^* := \underline{z}, \liminf_{t\to+\infty} y(t) \geq v^* := \underline{y}.$$

Theorem 2.3 and the above arguments imply that, if conditions (i) and (ii) of Theorem 2.5 hold, then the system (2.2) is uniformly persistent. □

Theorem 2.6. *If* $\alpha e^{-d_1\tau} - d_2 < 0$ *holds, then system* (2.2) *is not persistent.*

Proof. It follows from the first equation of system (2.2) that

$$\dot{x}_2(t) \leq \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t). \tag{2.9}$$

Consider the following auxiliary equation

$$\dot{u}(t) = \alpha e^{-d_1\tau} u(t-\tau) - d_2 u(t) - d_3 u^2(t). \tag{2.10}$$

By Lemma 2.2, if $\alpha e^{-d_1\tau} - d_2 < 0$, then it follows that

$$\lim_{t\to+\infty} u(t) = 0.$$

By comparison, we derive that

$$\lim_{t\to+\infty} x_2(t) = 0.$$

Then there is a $T > 0$, such that for any sufficiently small $\varepsilon > 0$ and $\varepsilon < m$, we have $x_2(t) < \varepsilon$ for $t > T$. From the second equation of (2.2), we have

$$\dot{z}(t) < \frac{\delta\varepsilon}{m} - \delta z(t). \tag{2.11}$$

Consider the following equation

$$\dot{u}(t) = \frac{\delta\varepsilon}{m} - \delta u(t). \tag{2.12}$$

Clearly, if $\frac{\delta\varepsilon}{m} < \delta$, then by Lemma 2.1, we have

$$\lim_{t\to+\infty} u(t) = \frac{\varepsilon}{m}.$$

Setting $\varepsilon \to 0$, then

$$\lim_{t\to+\infty} u(t) = 0.$$

Hence, by comparison, we have

$$\lim_{t\to+\infty} z(t) = 0.$$

Consider the last equation of system (2.2)

$$\dot{y}(t) = -d_4 y(t) + hz(t). \tag{2.13}$$

By the same method, we have

$$\lim_{t\to+\infty} y(t) = 0.$$

This proves the theorem. □

By setting $\dot{x}_2(t) = \dot{z}(t) = \dot{y}(t) = 0$ in system (2.2), it is easy to see that system (2.2) has at least one equilibrium $E_0(0,0,0)$. Because system (2.2) cannot be linearized, the local stability of E_0 will not be studied. We leave it as a future problem. If

$$\alpha e^{-d_1\tau} - d_2 > 0,$$

then the system (2.2) has a nonnegative equilibrium $E_1(x_2^0, 0, 0)$, where

$$x_2^0 = \frac{\alpha e^{-d_1\tau} - d_2}{d_3}. \tag{2.14}$$

It is easy to see that if (i) $h > d_4$ and (ii) $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$ hold, then system (2.2) has a unique positive equilibrium $E^*(x_2^*, z^*, y^*)$, where

$$x_2^* = \frac{1}{d_3}\left[\alpha e^{-d_1\tau} - d_2 - \frac{p}{m}\left(1 - \frac{d_4}{h}\right)\right],$$

$$z^* = \frac{1}{m}\left(1 - \frac{d_4}{h}\right)x_2^*,$$

$$y^* = \frac{1}{m}\left(\frac{h}{d_4} - 1\right)x_2^*.$$

3. Global Asymptotic Stability of E_1

In this section, we concentrate on the study of the stability of E_1. Hence, we assume $\alpha e^{-d_1\tau} - d_2 > 0$ holds, then the equilibrium E_1 of (2.2) exists. By our study, we obtain the global stability of E_1.

Now, we introduce the following results (quoted in many papers such as in [29]) which may be found in [26] and used to study the global stability of E_1.

Consider the following ordinary differential systems

$$\dot{x} = f(t, x), \tag{3.1}$$

$$\dot{y} = g(y), \tag{3.2}$$

where f and g are continuous and locally Lipschitz in $x \in R^n$ and solutions exist for all $t > 0$. System (3.1) is called asymptotically autonomous with limit system (3.2) if $f(t, x) \to g(x)$ as $t \to +\infty$ uniformly for $x \in R^n$.

Lemma 3.1. *Let e be a locally asymptotically stable equilibrium of* (3.2) *and* ω *be the* $\omega - $*limit set of a forward bounded solution* $x(t)$ *of* (3.1). *If* ω *contains a point* y_0 *such that the solution of* (3.2), *with* $y(0) = y_0$ *converges to e as* $t \to +\infty$, *then* $\omega = \{e\}$, *i.e.* $x(t) \to e$ *as* $t \to +\infty$.

Corollary 3.2. *If solutions of the system* (3.1) *are bounded and the equilibrium e of the limit system* (3.2) *is globally asymptotically stable, then any solution* $x(t)$ *of the system* (3.1) *satisfies* $x(t) \to e$ *as* $t \to \infty$.

Theorem 3.3. *Assume* $\alpha e^{-d_1\tau} - d_2 > 0$. *If* $d_4 > h$, *then* E_1 *is globally asymptotically stable.*

Proof. Linearizing system (2.2) at $E_1 = (x_2^0, 0, 0)$, we derive the characteristic equation of the equilibrium $E_1 = (x_2^0, 0, 0)$ is

$$(\lambda + 2\alpha e^{-d_1\tau} - d_2 - \alpha e^{-\lambda\tau} e^{-d_1\tau})[\lambda^2 + (d_4 + \delta)\lambda + \delta d_4 - h\delta] = 0, \tag{3.3}$$

Obviously, if $d_4 < h$, equation

$$\lambda^2 + (d_4 + \delta)\lambda + \delta d_4 - h\delta = 0 \tag{3.4}$$

has a positive real root; if $d_4 > h$, then (3.4) has two negative real part; All other eigenvalues are determined by the solutions of equation

$$\lambda + 2\alpha e^{-d_1\tau} - d_2 - \alpha e^{-\lambda\tau} e^{-d_1\tau} = 0. \tag{3.5}$$

If $\alpha e^{-d_1\tau} - d_2 > 0$, we will show that all of the eigenvalue of (3.5) have negative real part. Suppose that $\mathrm{Re}\,\lambda \geq 0$, then it follows from (3.5) that

$$\begin{aligned} \mathrm{Re}\,\lambda &= -2\alpha e^{-d_1\tau} + d_2 + \alpha e^{-d_1\tau} e^{-\tau \mathrm{Re}\,\lambda} \cos(\tau Im\lambda), \\ &< -2\alpha e^{-d_1\tau} + d_2 + \alpha e^{-d_1\tau}, \\ &= -\alpha e^{-d_1\tau} + d_2 < 0. \end{aligned} \tag{3.6}$$

It is a contradiction. Thus we have $\mathrm{Re}\,\lambda < 0$.

Hence, if $\alpha e^{-d_1\tau} - d_2 > 0$, and $d_4 < h$, then one of the eigenvalues of (3.3) has positive real part. Therefore, E_1 is unstable.

If $\alpha e^{-d_1\tau} - d_2 > 0$ and $d_4 > h$, then all of the eigenvalues of (3.3) have negative real parts. Therefore, E_1 is locally asymptotically stable.

In the following, we show the nonnegative equilibrium E_1 is globally asymptotically stable, if $\alpha e^{-d_1\tau} - d_2 > 0$ and $d_4 > h$.

From the last two equations of Eq. (2.2), we have

$$\begin{cases} \dot{z}(t) < \delta y(t) - \delta z(t), \\ \dot{y}(t) = -d_4 y(t) + hz(t). \end{cases}$$

Consider the following equations

$$\begin{cases} \dot{u}(t) = \delta v(t) - \delta u(t), \\ \dot{v}(t) = -d_4 v(t) + hz(t). \end{cases} \tag{3.7}$$

Obviously, when $d_4 > h$, then the unique equilibrium $(0, 0)$ of Eq. (3.7) exists and is locally and globally asymptotically stable. By comparison, we get

$$\lim_{t\to+\infty} z(t) = 0, \quad \lim_{t\to+\infty} y(t) = 0,$$

thus, for an arbitrary positive number ε small enough, there exists a time T, such that $y(t) < \varepsilon$, for all $t > T$. Then by the first equation of (2.2), we have

$$\begin{aligned} \dot{x}_2(t) &= \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - \frac{p x_2(t) y(t)}{m y(t) + x_2(t)} \\ &\geq \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t) - p y(t). \end{aligned} \tag{3.8}$$

Considering the solutions of equation

$$\dot{u}(t) = \alpha e^{-d_1\tau} u(t-\tau) - d_2 u(t) - d_3 u^2(t) - p y(t). \tag{3.9}$$

The limit equation of (3.9) is

$$\dot{u}(t) = \alpha e^{-d_1\tau} u(t-\tau) - d_2 u(t) - d_3 u^2(t). \tag{3.10}$$

By Lemma 2.2 and $\alpha e^{-d_1\tau} - d_2 > 0$, we get the solutions of (3.10) is close to x_2^0, as $t \to \infty$, that is $\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-d_1\tau} - d_2}{d_3} = x_2^0$. It follows from Corollary 3.2 that the solutions of (3.9) satisfies $\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-d_1\tau} - d_2}{d_3} = x_2^0$. By compare, we have

$$\lim_{t\to+\infty} x_2(t) \geq \frac{\alpha e^{-d_1\tau} - d_2}{d_3} = x_2^0.$$

Moreover, it is obvious that

$$\dot{x}_2(t) \leq \alpha e^{-d_1\tau} x_2(t-\tau) - d_2 x_2(t) - d_3 x_2^2(t).$$

Clearly,

$$\lim_{t\to+\infty} x_2(t) \leq \frac{\alpha e^{-d_1\tau} - d_2}{d_3} = x_2^0.$$

Then, it is easy to see that

$$\lim_{t\to+\infty} x_2(t) = \frac{\alpha e^{-d_1\tau} - d_2}{d_3} = x_2^0.$$

This completes the proof. □

4. Stability and Hopf Bifurcation at the Equilibrium E^*

In this section, we are going to study the stability of $E^*(x_2^*, z^*, y^*)$. Hence, we assume throughout this section that (i) $h > d_4$ and (ii) $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$ hold, which ensures the existence of this steady-state.

Linearizing system (2.2) at $E^*(x_2^*, z^*, y^*)$, we derive the characteristic equation of the equilibrium $E^*(x_2^*, z^*, y^*)$ is

$$\left[\lambda + d_2 + 2d_3x_2^* + \frac{pmy^{*^2}}{(my^* + x_2^*)^2} - \alpha e^{-\lambda\tau}e^{-d_1\tau}\right]\left[\lambda^2 + (d_4+\delta)\lambda + \delta d_4 - \frac{h\delta x_2^{*^2}}{(my^* + x_2^*)^2}\right]$$
$$+ \frac{\delta my^{*^2}hpx_2^{*^2}}{(my^* + x_2^*)^4} = 0. \tag{4.1}$$

Equation (4.1) can be written as

$$P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{4.2}$$

where

$$P(\lambda,\tau) = \lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau), \tag{4.3}$$
$$Q(\lambda,\tau) = B_2(\tau)\lambda^2 + B_1(\tau)\lambda + B_0(\tau), \tag{4.4}$$

and

$$A_2(\tau) = d_4 + \delta + G(\tau),$$
$$A_1(\tau) = G(\tau)(d_4+\delta) + \delta d_4 - \frac{h\delta x_2^{*^2}}{(my^* + x_2^*)^2}$$
$$= (d_4+\delta)G(\tau) + \delta d_4\left(1 - \frac{d_4}{h}\right),$$
$$A_0(\tau) = G(\tau)\left(\delta d_4 - \frac{h\delta x_2^{*^2}}{(my^* + x_2^*)^2}\right) + \frac{\delta my^{*^2}hpx_2^{*^2}}{(my^* + x_2^*)^4}$$
$$= \delta d_4\left(1 - \frac{d_4}{h}\right)\left[G(\tau) + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right],$$

$$
\begin{aligned}
B_2(\tau) &= -\alpha e^{-d_1\tau},\\
B_1(\tau) &= -\alpha e^{-d_1\tau}(d_4+\delta),\\
B_0(\tau) &= -\alpha e^{-d_1\tau}\delta d_4\left(1-\frac{d_4}{h}\right),\\
G(\tau) &= d_2 + 2d_3x_2^* + \frac{pmy^{*^2}}{(my^*+x_2^*)^2}\\
&= 2\alpha e^{-d_1\tau} - d_2 - \frac{2p}{m}\left(1-\frac{d_4}{h}\right) + \frac{p}{m}\left(1-\frac{d_4}{h}\right)^2.
\end{aligned}
\tag{4.5}
$$

Obviously $G(\tau) > 0$, and clearly $\lambda = 0$ is not a root of (4.2), since

$$
A_0(\tau)+B_0(\tau) = \delta d_4\left(1-\frac{d_4}{h}\right)\left[G(\tau)-\alpha e^{-d_1\tau}+\frac{pd_4}{mh}\left(1-\frac{d_4}{h}\right)\right] \neq 0.
$$

When $\tau = 0$, Eq. (4.2) reduces to

$$
\lambda^3 + (A_2(0)+B_2(0))\lambda^2 + (A_1(0)+B_1(0))\lambda + A_0(0)+B_0(0) = 0. \tag{4.6}
$$

Note that

$$
\begin{aligned}
A_2(0)+B_2(0) &= d_4+\delta+G(0)-\alpha,\\
A_1(0)+B_1(0) &= (d_4+\delta)(G(0)-\alpha)+\delta d_4\left(1-\frac{d_4}{h}\right),\\
A_0(0)+B_0(0) &= \delta d_4\left(1-\frac{d_4}{h}\right)\left[G(0)-\alpha+\frac{pd_4}{mh}\left(1-\frac{d_4}{h}\right)\right],\\
&(A_2(0)+B_2(0))(A_1(0)+B_1(0)) - (A_0(0)+B_0(0))\\
&= (d_4+\delta)\left[(d_4+\delta)(G(0)-\alpha)+\delta d_4\left(1-\frac{d_4}{h}\right)\right]\\
&\quad + (d_4+\delta)(G(0)-\alpha)^2 + \frac{pd_4^2\delta}{mh}\left(1-\frac{d_4}{h}\right)^2.
\end{aligned}
\tag{4.7}
$$

By Routh–Hurwitz Theorem, from which it is easy to know that all characteristic roots of (4.6) have negative real parts if and only if

$$
G(0)-\alpha+\frac{pd_4}{mh}\left(1-\frac{d_4}{h}\right) > 0
$$

holds. Then we have the following theorem.

Theorem 4.1. *Assume* (i) $h > d_4$ *and* (ii) $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$ *hold. When* $\tau = 0$, *then the unique positive equilibrium* $E^*(x_2^*, z^*, y^*)$ *of system* (2.2) *is locally asymptotically stable if and only if*

$$G(0) - \alpha + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right) > 0. \tag{4.8}$$

In the following, we study the existence of purely imaginary roots $\lambda = i\omega$, $\omega \in R$. Equation (4.2) takes the form of a third-degree exponential polynomial in λ, in which all the coefficients of P and Q are dependent on τ. Beretta and Kuang [6] established a geometrical criterion which gives the existence of purely imaginary roots for a characteristic equation with delay dependent coefficients. In order to apply the criterion due to Beretta and Kuang [6], we need to verify the following properties, for all $\tau \in [0, \tau_{\max})$, where $\tau_{\max}$ is the maximum value which E^* exists.

(i) $P(0, \tau) + Q(0, \tau) \neq 0$;
(ii) $P(i\omega, \tau) + Q(i\omega, \tau) \neq 0$;
(iii) $\limsup\{\|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}|; |\lambda| \to \infty, \ \mathrm{Re}\,\lambda \geq 0\} < 1$;
(iv) $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ for each τ has at most a finite number of real zeros;
(v) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ are defined as in (4.3) and (4.4).

Properties (i), (ii) and (iii) can be easily verified. Let $\tau \in [0, \tau_{\max}]$, it is easy to see that

$$P(0, \tau) + Q(0, \tau) = A_0(\tau) + B_0(\tau) \neq 0.$$

Moreover,

$$\begin{aligned} P(i\omega, \tau) + Q(i\omega, \tau) &= -i\omega^3 - A_2\omega^2 + iA_1\omega - B_2\omega^2 + iB_1\omega + B_0, \\ &= [-(A_2(\tau) + B_2(\tau))\omega^2 + A_0(\tau) + B_0(\tau)] \\ &\quad + i[-\omega^3 + (A_1(\tau) + B_1(\tau)\omega)]. \end{aligned} \tag{4.9}$$

Hence, (ii) is true. It is easy to get $|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}| \sim |\frac{B_2(\tau)}{\lambda}|$. Therefore, (iii) is also true. From (4.3) and (4.4), we have

$$\begin{aligned} |P(i\omega, \tau)|^2 &= \omega^6 + [A_2^2 - 2A_1]\omega^4 + [A^2 - 2A_2A_0]\omega^2 + A_0^2, \\ |Q(i\omega, \tau)|^2 &= B_2^2\omega^4 + [B_1^2 - 2B_2B_0]\omega^2 + B_0^2. \end{aligned}$$

Then we have

$$F(\omega, \tau) = \omega^6 + p(\tau)\omega^4 + q(\tau)\omega^2 + r(\tau),$$

where

$$\begin{aligned} p(\tau) &= A_2^2(\tau) - 2A_1(\tau) - B_2^2(\tau), \\ q(\tau) &= A_1^2(\tau) - 2A_2(\tau)A_0(\tau) + 2B_0(\tau)B_2(\tau) - B_1^2(\tau), \\ r(\tau) &= A_0^2(\tau) - B_0^2(\tau). \end{aligned} \tag{4.10}$$

Straightforward calculations show

$$\begin{aligned} p(\tau) &= [G(\tau)]^2 + \delta^2 + d_4^2 + \frac{2\delta d_4^2}{h} - (\alpha e^{-d_1\tau})^2, \\ q(\tau) &= \left(\delta^2 + d_4^2 + 2\frac{\delta d_4^2}{h}\right)\{[G(\tau)]^2 - (\alpha e^{-d_1\tau})^2\} + \left[\delta d_4\left(1 - \frac{d_4}{h}\right)\right]^2 \\ &\quad - 2\frac{\delta p d_4^2}{mh}\left(1 - \frac{d_4}{h}\right)^2 [\delta + d_4 + G(\tau)], \\ r(\tau) &= \left[\delta d_4\left(1 - \frac{d_4}{h}\right)\right]^2 \left[\alpha e^{-d_1\tau} + G(\tau) + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right] \\ &\quad \times \left[G(\tau) - \alpha e^{-d_1\tau} + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right]. \end{aligned} \tag{4.11}$$

It is obvious that property (iv) is satisfied. Let (ω_0, τ_0) be a point of its domain such that $F(\omega_0, \tau_0) = 0$. Obviously, the partial derivatives F_ω and F_τ exist and continuous in a certain neighborhood of (ω_0, τ_0) and $F_\omega(\omega_0, \tau_0) \neq 0$. By Implicit Function Theorem, (v) is also satisfied.

Now assume that $\lambda = i\omega (\omega > 0)$ is a purely imaginary characteristic root of (4.2). Substituting it into Eqs. (4.3) and (4.4), separating real and imaginary parts yields

$$-A_1(\tau)\omega^2 + A_0(\tau) = -[-B_2(\tau) + B_0(\tau)]\cos(\omega\tau) - B_1(\tau)\omega\sin(\omega\tau), \tag{4.12}$$

$$-\omega^3 + A_1(\tau)\omega = [-B_1(\tau)]\omega\cos(\omega\tau) + [-B_2(\tau)\omega^2 + B_0(\tau)]\sin(\omega\tau). \tag{4.13}$$

It is easy to check that if (ω, τ) is a solution of Eqs. (4.12) and (4.13), then so is $(-\omega, \tau)$. Hence, if $i\omega$ is a purely imaginary characteristic root of Eqs. (4.12) and (4.13), its conjugate has the same property. In the following, we only look for purely imaginary roots of Eqs. (4.12) and (4.13) with positive imaginary part. Squaring both sides of (4.12) and (4.13) and adding them up, we get

$$\omega^6 + p(\tau)\omega^4 + q(\tau)\omega^2 + r(\tau) = 0. \tag{4.14}$$

That is $F(\omega, \tau) = 0$.

Let $x = \omega^2$. Then (4.14) becomes

$$h(x) := x^3 + p(\tau)x^2 + q(\tau)x + r(\tau) = 0. \tag{4.15}$$

We denote

$$\Delta(\tau) = p^2(\tau) - 3q(\tau), \tag{4.16}$$

and, when $\Delta(\tau) \geq 0$,

$$x_0(\tau) = \frac{-p(\tau) + \sqrt{\Delta(\tau)}}{3}, \tag{4.17}$$

then we have the following lemma (details of the proof are given in Ruan and Wei [21], Lemma 2.2).

Lemma 4.2. *Let $\tau \in [0, \tau_{\max})$ and $\Delta(\tau)$, $x_0(\tau)$ be defined by (4.16) and (4.17), respectively. Then $h(x)$, defined in (4.15), has positive roots if and only if $r(\tau) < 0$ or $r(\tau) \geq 0$, $\Delta(\tau) \geq x_0(\tau) > 0$ and $h(x_0(\tau), \tau) < 0$.*

Note that

$$\begin{aligned} r(\tau) = {} & \left[\delta d_4 \left(1 - \frac{d_4}{h}\right)\right]^2 \left[\alpha e^{-d_1\tau} + G(\tau) - \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right] \\ & \times \left[G(\tau) - \alpha e^{-d_1\tau} + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right], \end{aligned}$$

then we get a sufficient condition for $r(\tau) < 0$:

$$\left[\alpha e^{-d_1\tau} + G(\tau) - \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right]\left[G(\tau) - \alpha e^{-d_1\tau} + \frac{pd_4}{mh}\left(1 - \frac{d_4}{h}\right)\right] < 0.$$

Consequently, $r(0) < 0$ and considering the continuity of $r(\tau)$, we deduce that there exists $\bar{\tau} > 0$, such that $r(\tau) < 0$ for $\tau \in [0, \bar{\tau})$. Set $I := [0, \bar{\tau})$. Then there exists $\tau \in [0, \bar{\tau})$ such that $F(\omega(\tau), \tau) = 0$.

From (4.12) and (4.13), we get

$$\cos(\omega\tau) = \frac{(B_1 - A_2B_2)\omega^4 + (A_2B_0 + A_0B_2 - A_1B_1)\omega^2 - A_0B_0}{B_2^2\omega^4 + (B_1^2 - 2B_2B_0)\omega^2 + B_0^2}, \tag{4.18}$$

$$\sin(\omega\tau) = \frac{B_2\omega^5 + (A_2B_1 - A_1B_2 - B_0)\omega^3 + (A_1B_0 - A_0B_1)\omega}{B_2^2\omega^4 + (B_1^2 - 2B_2B_0)\omega^2 + B_0^2}, \tag{4.19}$$

where we deliberately omit the dependence of the parameter on τ.

Define the function $\theta(\tau) \in [0, 2\pi]$, and $\cos(\theta\tau)$ and $\sin(\theta\tau)$ are given by the right hands of (4.18) and (4.19), respectively, such that $\omega(\tau) = \theta(\tau) + 2n\pi$, $n = 0, 1, 2, \ldots$. Hence, we define the maps:

$$\tau_n = \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n = 0, 1, 2, \ldots,$$

where $\omega(\tau)$ is a positive root of (4.14). Using (4.12) and (4.13), we can verify that $i\omega^*$ with $\omega^* = \omega(\tau^*)$ is a root of (4.14) if and only if τ^* is a root of function S_n, defined by

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau \in I, \ n = 0, 1, 2, \ldots.$$

The following theorem is due to Beretta and Kuang [6].

Theorem 4.3. *Assume that $\omega(\tau)$ is a positive root of (4.14) defined for $\tau \in I$, and at some $\tau^* \in I$,*

$$S_n(\tau^*) = 0$$

for some $n = 0, 1, 2, \ldots$. Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega(\tau^*)$ of (4.2) exists at $\tau = \tau^*$, which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$, and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where

$$\delta(\tau^*) = \mathrm{sign}\left\{ \left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right|_{\lambda = i\omega(\tau^*)} \right\} = \mathrm{sign}\{F_\omega(\omega(\tau^*, \tau^*))\}\,\mathrm{sign}\left\{ \left.\frac{ds_n(\tau)}{d\tau}\right|_{\tau=\tau^*} \right\}. \quad (4.20)$$

Since $\frac{\partial F}{\partial \omega}(\omega, \tau) = 2\omega \frac{\partial h}{\partial x}(\omega^2, \tau)$, condition (4.20) can be rewritten as

$$\delta(\tau^*) = \mathrm{sign}\left\{ \left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right|_{\lambda = i\omega(\tau^*)} \right\} = \mathrm{sign}\frac{\partial h}{\partial x}(\omega^2(\tau^*), \tau^*)\,\mathrm{sign}\left\{ \left.\frac{ds_n(\tau)}{d\tau}\right|_{\tau=\tau^*} \right\}. \quad (4.21)$$

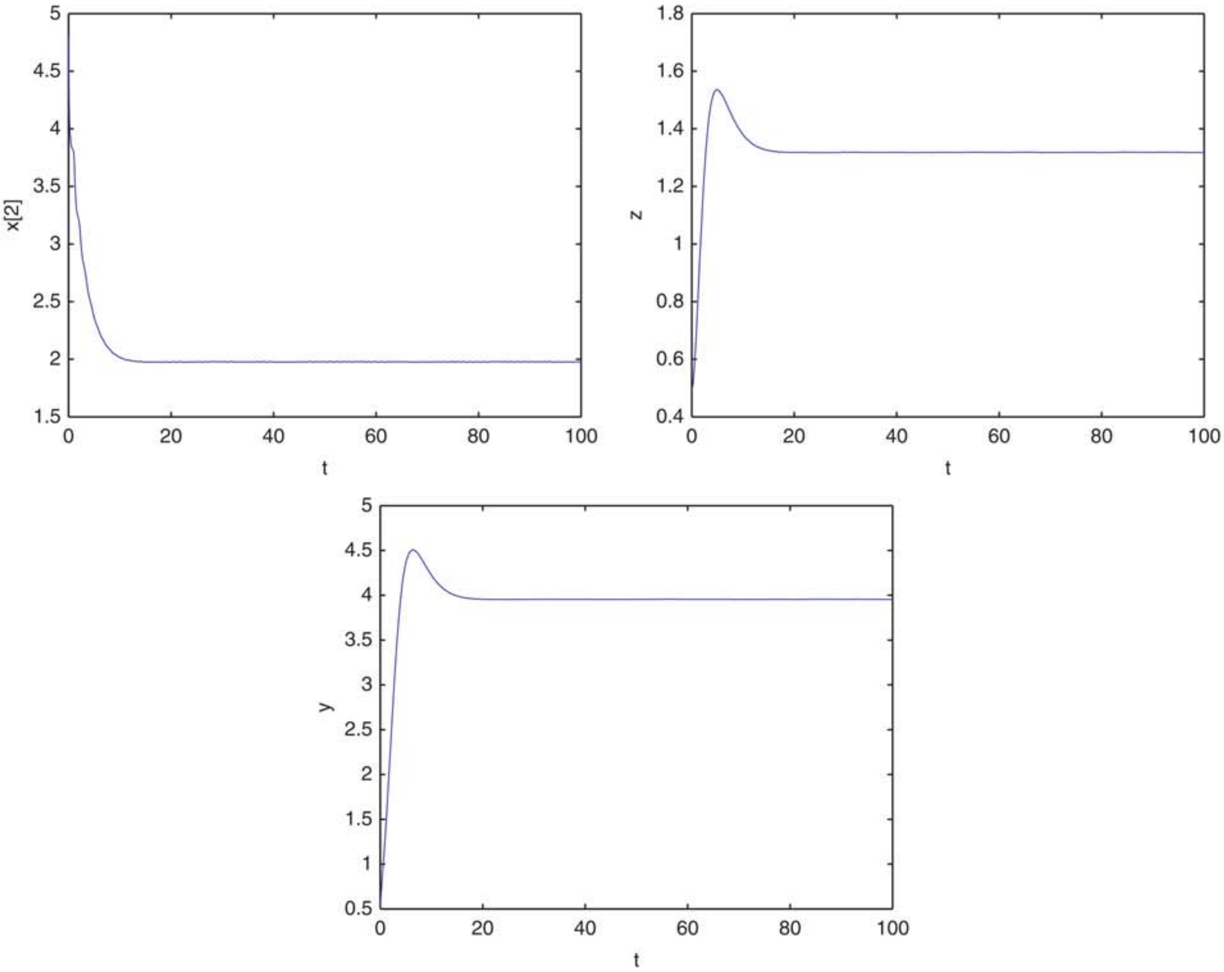

Fig. 1. The temporal solution found by numerical simulation of system (2.2) with $\alpha = 10$, $d_1 = 1, d_2 = e^{-1}, d_3 = 1, p = 2, m = 1, d_4 = 1, \delta = 1.5, h = 3, \tau = 1$ and initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (4.8, 0.5, 0.5)$.

It can be easily observed that $S_n(0) < 0$. Moreover, for all $\tau \in I$, $S_n(\tau) > S_{n+1}(\tau)$, with $n \in N$. Hence, if $S_0(\tau) < 0$ has no root in I, then $S_n(\tau)$ has no root in I. If $S_n(\tau)$ has a positive roots $\tau \in I$ for some with $n \in N$, then there is at least one root satisfying

$$\frac{dS_n(\tau)}{d\tau} > 0.$$

Applying Theorem 4.1, we can conclude the existence of a Hopf bifurcation as stated in the following theorem.

Theorem 4.4. *For system* (2.2), *assume* (i) $h > d_4$ *and* (ii) $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$, *condition* (4.8) *hold true.*

(i) *If the function* $S_0(\tau)$ *has no positive in* I, *then the equilibrium* E^* *is locally asymptotically stable for all* $\tau \geq 0$,
(ii) *If the function* $S_0(\tau)$ *has a positive root in* I, *then there at least exists a* $\tau^* \in I$, *such that the positive equilibrium* E^* *is locally asymptotically stable*

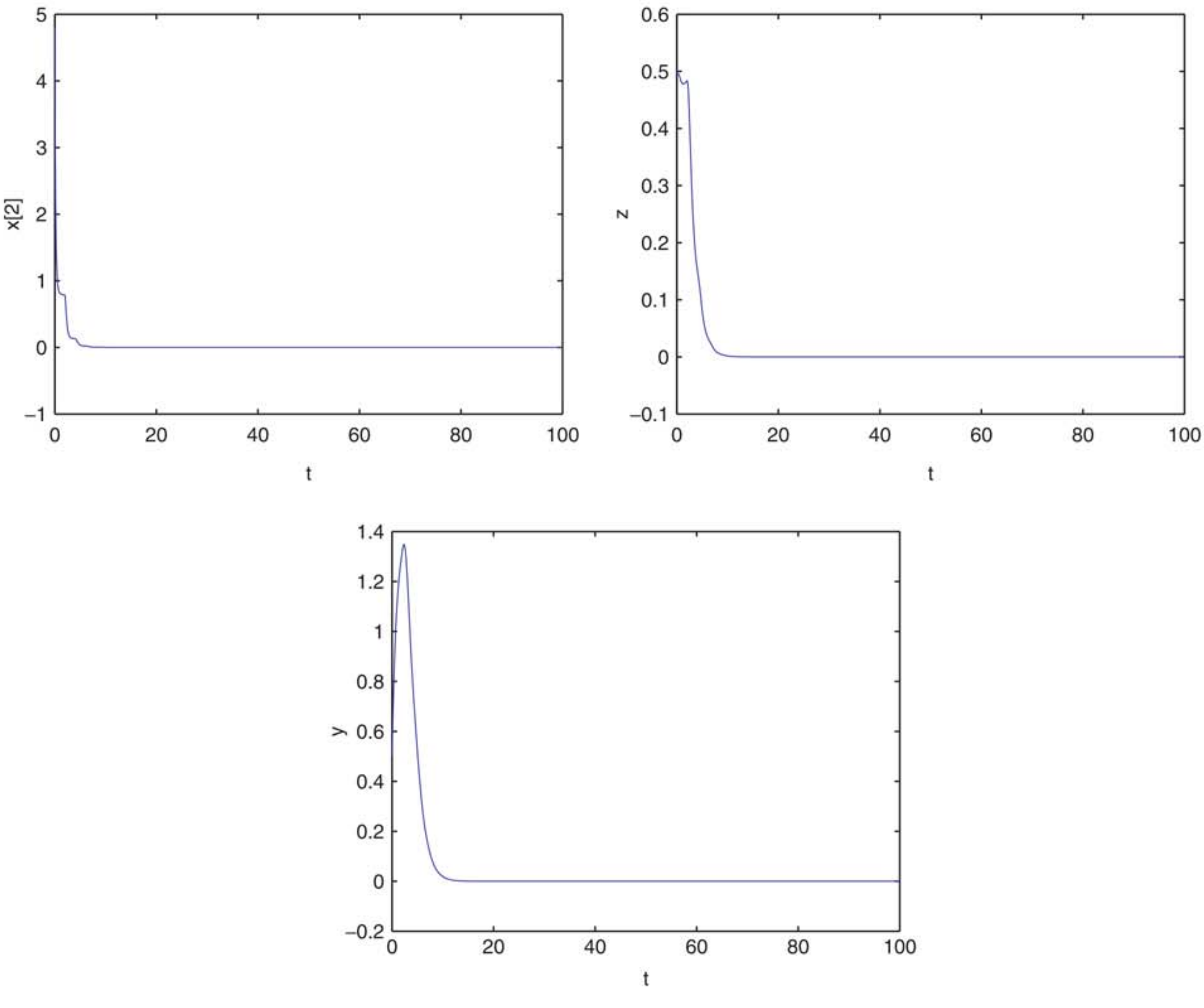

Fig. 2. The temporal solution found by numerical simulation of system (2.2) with $\alpha = 6$, $d_1 = 1, d_2 = 8e^{-1}, d_3 = 1, p = 2, m = 1, d_4 = 1, \delta = 1.5, h = 3, \tau = 2$ and initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (4.8, 0.5, 0.5)$.

for $0 \le \tau < \tau^*$ and becomes unstable for $\tau \ge \tau^*$, with a Hopf bifurcation occurring when $\tau = \tau^*$, if and only if

$$\frac{\partial h}{\partial x}(\omega^2(\tau^*), \tau^*) < 0.$$

5. Numerical Simulations

In this section, we shall carry out some numerical simulations for supporting our theoretical analysis.

(1) In system (2.2), we select $\alpha = 10, d_1 = 1, d_2 = e^{-1}, d_3 = 1, p = 2, m = 1, d_4 = 1, \delta = 1.5, h = 3, \tau = 1$. It is easy to show that (i) $\alpha e^{-d_1\tau} - d_2 - \frac{p}{m} \ge 0$ and (ii) $h \ge d_4$. By Theorem 2.6, we see that system (2.2) is permanent. Numerical simulation illustrates our result (see Fig. 1).

(2) In system (2.2), we select $\alpha = 6, d_1 = 1, d_2 = 8e^{-1}, d_3 = 1, p = 2, m = 1, d_4 = 1, \delta = 1.5, h = 3, \tau = 2$. Clearly, $\alpha e^{-d_1\tau} - d_2 < 0$. By Theorem 2.7, we

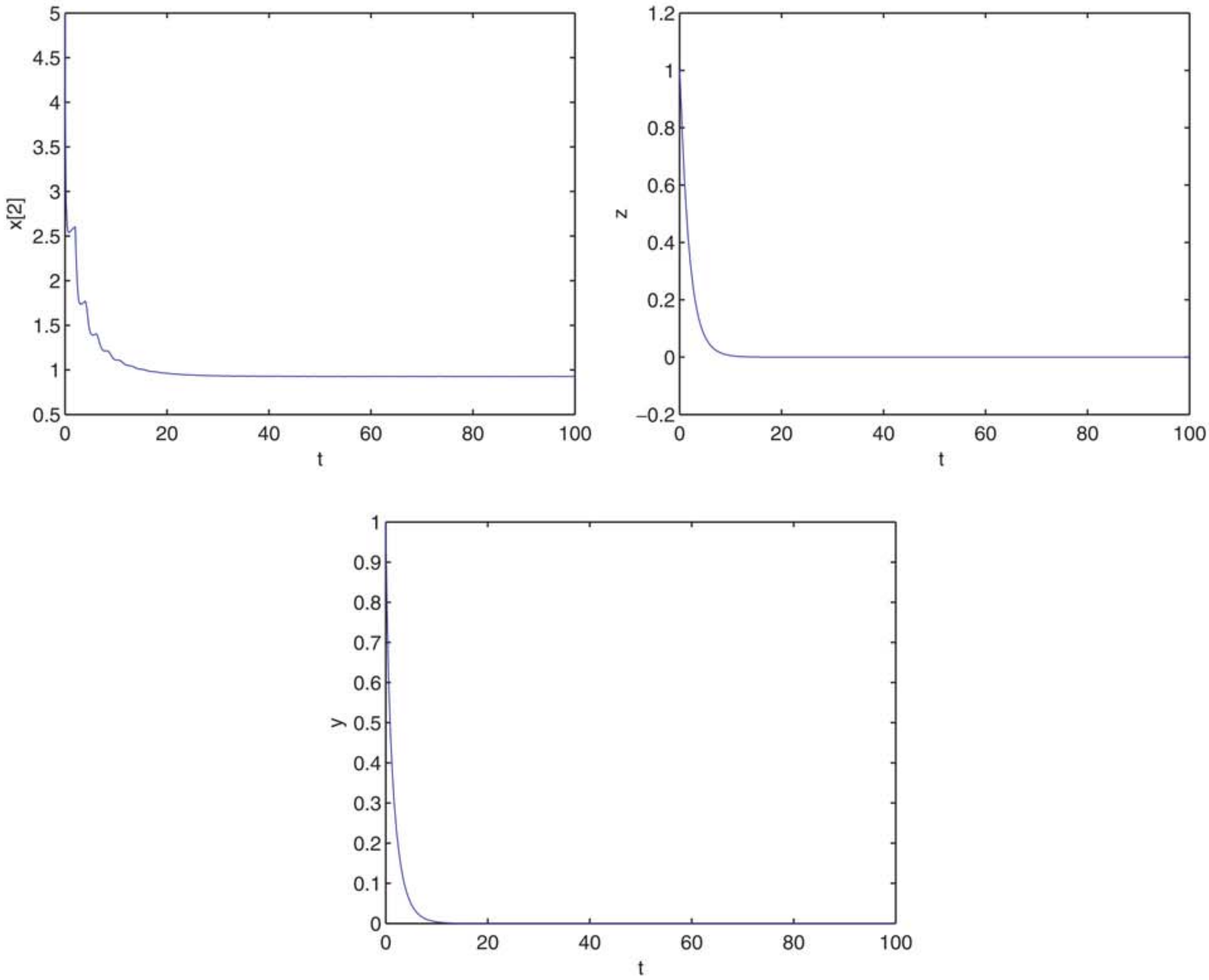

Fig. 3. The temporal solution found by numerical simulation of system (2.2) with $\alpha = 15, d_1 = 1, d_2 = 3e^{-1}, d_3 = 1, p = 2, m = 1, d_4 = 2, \delta = 1.5, h = 1, \tau = 2$ and initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (5, 1, 1)$.

see that the system (2.2) will be extinct. Numerical simulation illustrates our result (see Fig. 2).

(3) In system (2.2), we select $\alpha = 15, d_1 = 1, d_2 = e^{-3}, d_3 = 1, p = 2, m = 1, d_4 = 2, \delta = 1.5, h = 1, \tau = 2$. System (2.2) with above coefficients has a boundary equilibrium $E_1(0.9, 0, 0)$. Clearly, $\alpha e^{-d_1\tau} - d_2 > 0$ and $d_4 > h$. By Theorem 3.3, $E_1(0.9, 0, 0)$ is globally asymptotically stable. Numerical simulation illustrates our result (see Fig. 3).

(4) In system (2.2), we select $\alpha = 2, d_1 = 0.4, d_2 = 0.03, d_3 = 1, p = 2.2, m = 1, d_4 = 1, \delta = 0.3, h = 2, \tau = 0.73$. System (2.2) with above coefficients has a unique positive equilibrium $E^*(0.36, 0.18, 0.36)$. It is easy to show that $h > d_4$, $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$, By Theorem 4.4, the positive equilibrium E^* is stable. Numerical simulation illustrates our result (see Fig. 4).

(5) In system (2.2), we select $\alpha = 2, d_1 = 0.4, d_2 = 0.03, d_3 = 1, p = 2.2, m = 1, d_4 = 1, \delta = 0.3, h = 2, \tau = 0.8$. System (2.2) with above coefficients has a unique positive equilibrium $E^*(0.32, 0.16, 0.32)$. It is easy to show that $h > d_4$, $\alpha e^{-d_1\tau} - d_2 > \frac{p}{m}(1 - \frac{d_4}{h})$, and $\frac{\partial h}{\partial x}(\omega^2(\tau^*), \tau^*) > 0$, By Theorem 4.4, the positive

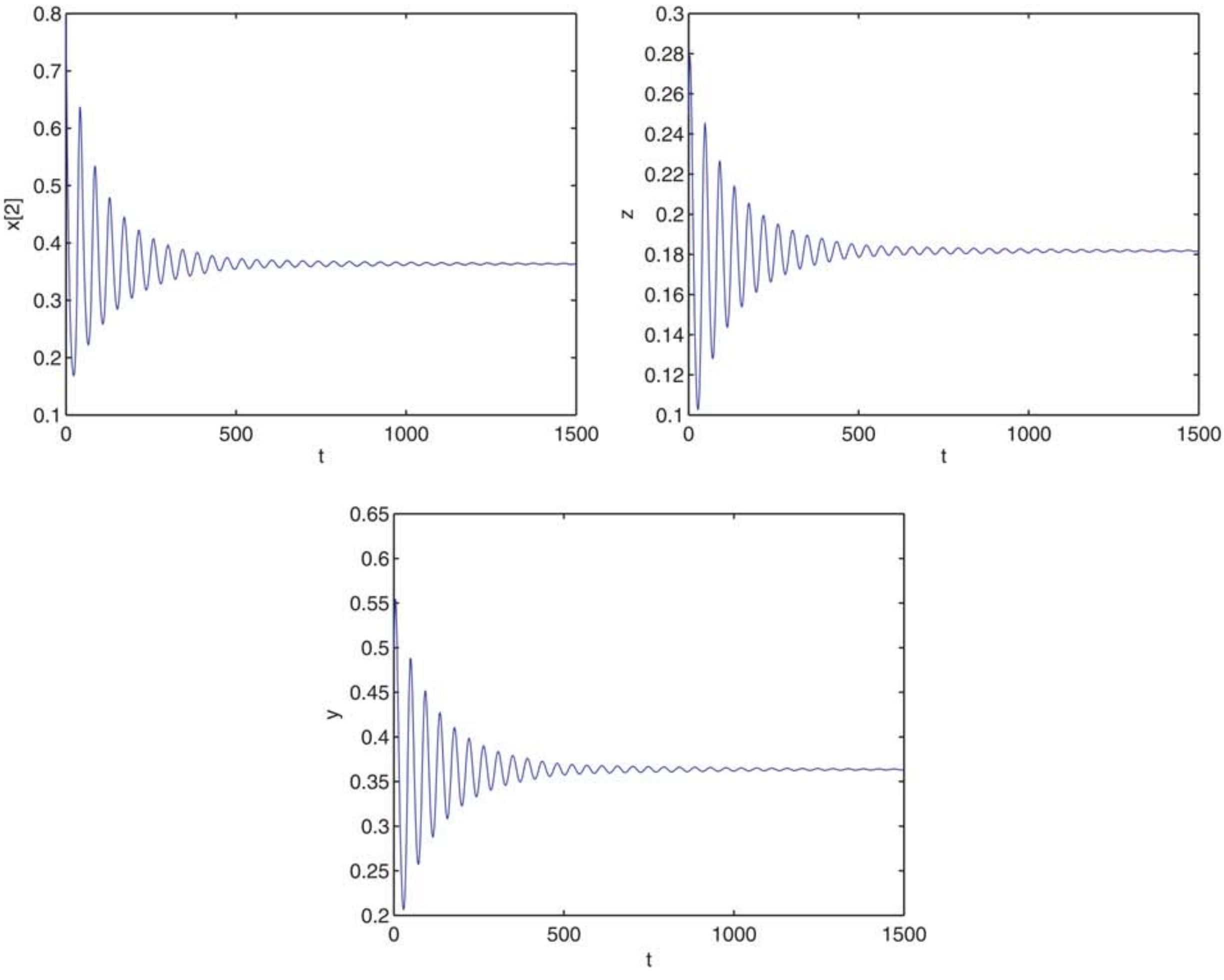

Fig. 4. The temporal solution found by numerical simulation of system (2.2) with $\alpha = 2$, $d_1 = 0.4, d_2 = 0.03, d_3 = 1, p = 2.2, m = 1, d_4 = 1, \delta = 0.3, h = 2, \tau = 0.73$ and initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (0.8, 0.25, 0.5)$.

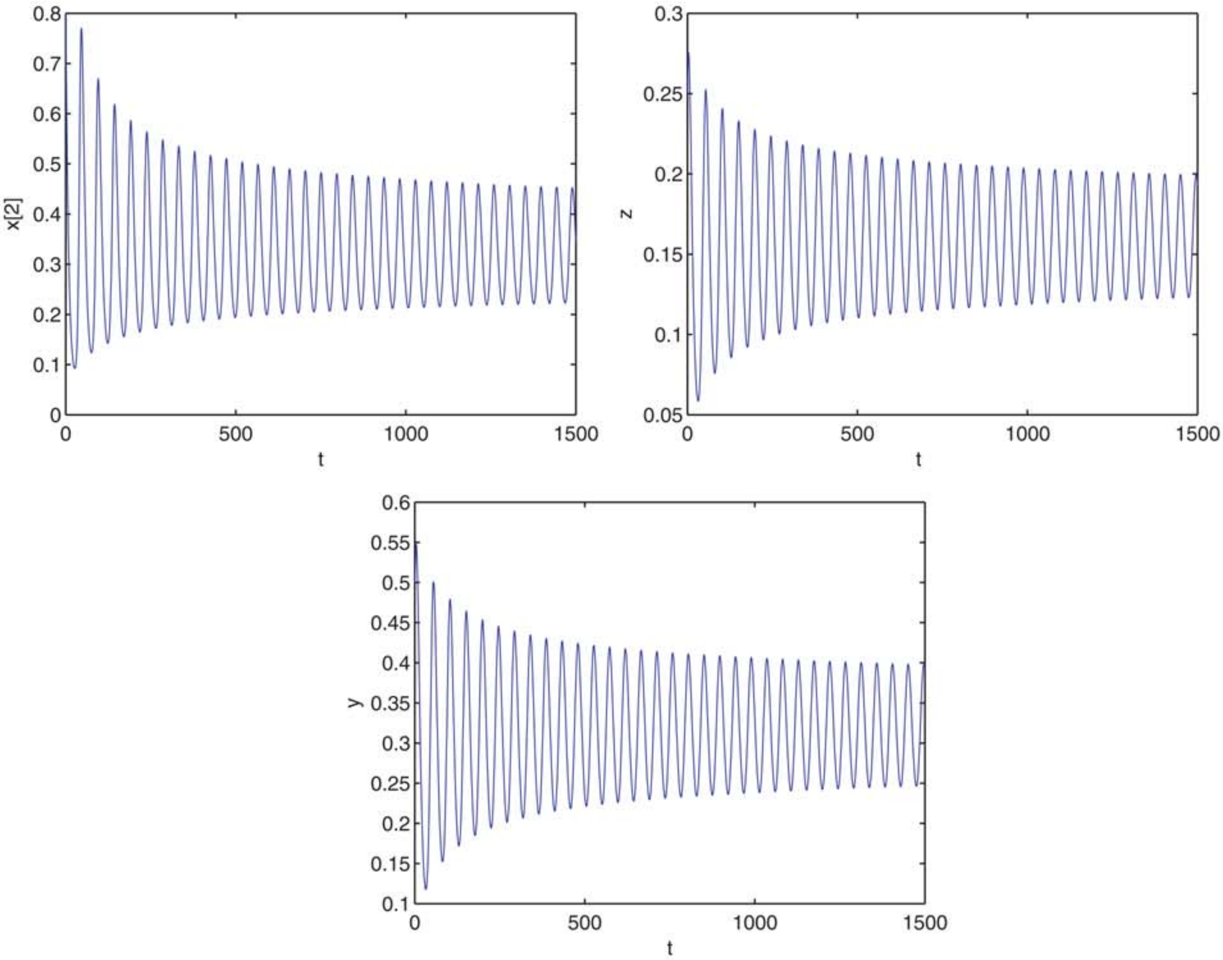

Fig. 5. The temporal solution found by numerical simulation of system (2.2) with $\alpha = 2$, $d_1 = 0.4, d_2 = 0.03, d_3 = 1, p = 2.2, m = 1, d_4 = 1, \delta = 0.3, h = 2, \tau = 0.8$ and initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (0.8, 0.25, 0.5)$.

equilibrium E^* is unstable and a Hopf bifurcation occurs, Numerical simulation illustrates our result (see Fig. 5).

6. Discussion

In this paper, a ratio-dependent predator–prey model with stage structure for prey and time delay is considered. We assume that the present level of the predator affects instantaneously the growth of the maturity prey, but that the growth of the predator is influenced by the amount of the maturity prey in the past. Then in Sec. 2, we give sufficient conditions for the permanence and extinction of the system. By Theorem 2.5, we derive that if $h > d_4$ and $0 \le \tau < \frac{1}{d_1} \ln \frac{m\alpha}{md_2+p}$ hold, then system (2.2) is uniformly persistent. By Theorem 2.6, we derive that if $\tau > \frac{1}{d_1} \ln \frac{\alpha}{d_2}$, then system (2.2) is not persistent. In Sec. 3, we assume $0 \le \tau < \frac{1}{d_1} \ln \frac{\alpha}{d_2}$ and $h < d_4$, then the nonnegative equilibrium E_1 exists. By Theorem 3.3, we know it is globally asymptotically stable for any time delay $0 \le \tau < \frac{1}{d_1} \ln \frac{\alpha}{d_2}$. In Sec. 4, we assume $h > d_4$ and $0 \le \tau < \frac{1}{d_1} \ln \frac{\alpha}{d_2 + \frac{p}{m}(1-\frac{d_4}{h})}$, which ensures the existence of the positive equilibrium E^* and then the stability of the positive equilibrium E^* is studied.

By Theorem 4.1, we know that for $\tau = 0$, $E^*(x_2^*, z^*, y^*)$ of system (2.2) is locally asymptotically stable, if and only if $G(0) - \alpha + \frac{pd_4}{mh}(1 - \frac{d_4}{h}) > 0$. By Theorem 4.4, we know that for $\tau \geq 0$, there will exist $\tau^* \in I$, such that the equilibrium E^* is asymptotically stable for $0 \leq \tau < \tau^*$, and becomes unstable for τ staying in some right neighborhood of τ^*, with a Hopf bifurcation occurring when $\tau = \tau^*$ if and only if $\frac{\partial h}{\partial x}(\omega^2(\tau^*), \tau^*) < 0$.

Figures 4 and 5 have the same parameter values: $\alpha = 2, d_1 = 0.4, d_2 = 0.03, d_3 = 1, p = 2.2, m = 1, d_4 = 1, \delta = 0.3, h = 2, \tau = 0.8$. Here initial value $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) = (0.8, 0.25, 0.5)$. In Fig. 4 the trajectory converges to the positive equilibrium at $\tau = 0.73$; Fig. 2 shows a periodic behavior at $\tau = 0.8$.

There are still many interesting and challenging mathematical questions that need to be studied. In this paper, we do not study the stability of the equilibrium $E_0(0, 0, 0)$. And we leave it as a future work.

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